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Subdiffusive transients in area-preserving mappings

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It is shown that for periodic area-preserving mappings with localized regular motion and global chaotic motion, the global diffusion has a long subdiffusive transient regime in the case of weak chaos. This is because the universal decay speed of boundary-layer trapping time distribution is only slightly greater than the critical value for asymptotic subdiffusion. Deviations from normal behavior are studied.

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Hamiltonian systems with $1\frac{1}{2}$ degrees of freedom are the lowest dimension Hamiltonian systems which may exhibit chaotic behavior. In spite of their relative simplicity, such systems provide models for many problems in various fields of physics. Some well known examples include magnetic confinement in tokamaks [1], particle motion in accelerators, and passive particle movement in three dimensional (3D) incompressible fluids [2,3]. For such Hamiltonian systems, there may exist in the phase space a global region to which a chaotic orbit is accessible. Transport properties of this chaotic region have both theoretic implications (such as those for statistical mechanics [4]) and practical applications (such as particle confinement in tokamaks [1]). Because of intermixing of strongly chaotic segments and quasiregular segments of the chaotic orbits, the "random walk" of a global chaotic orbit is not necessarily characterized by a Gaussian probability distribution. Our understanding of this issue is not conclusive and many recent studies are devoted to it [5–7]. According to dispersion rates of orbits in the phase space, $\langle r^2(t)\rangle \sim t^\gamma$, the diffusion is classified as normal with $\gamma = 1$, superdiffusive with $\gamma > 1$, or subdiffusive with $\gamma < 1$. A number of studies have been made for superdiffusion [7–9], but there is no conclusive answer to the existence of subdiffusion in similar systems. Although Schwägerl and Krug have interesting results on subdiffusion in a piecewise area-preserving map [10], their map is not analytic, and it is not clear whether their results are relevant to the analytic maps associated with Hamiltonian systems.

Hamiltonian systems with $1\frac{1}{2}$ degrees of freedom can be reduced to so-called area-preserving mappings. The latter contain essential properties of the former. In such systems, superdiffusion is associated with existence of unbounded regular orbits in the 3D phase space. Sticking of particles near the boundary of the regular motion region (named the KAM region after the Kolmogorov-Arnold-Moser theorem) creates long flights [2,7] and thus superdiffusion in the direction of the regular flow [8,11]. The existence of unbounded regular orbits in the Hamiltonian systems is reflected in the so-called accelerator modes in the corresponding area-preserving mapping. Accelerator modes are a series of regular (KAM) regions in phase space, which are extended to infinity by successively iterating the map. In a separate work, I show that the flights dominate diffusion; existence of accelerator modes inevitably gives rise to superdiffusion at asymptotically long times [11]. In the present investigation I restrict myself to area-preserving maps without accelerator modes. As a convenient example, I use the so-called fourfold stochastic web map, which has no accelerator modes for the parameters I consider.

The fourfold stochastic web map M is defined as

$$x_t = -y_{t-1} - a\sin(x_{t-1}),$$
$$y_t = x_{t-1}.$$

This map has 2π periodicity in both x and y directions. Unstable fixed points of M^4 form a square lattice of side length $\sqrt{2}\pi$. They are connected by a global chaotic region which looks like a network of channels for small a as shown in Fig. 1. As $a \to 0$ the thickness of the channels decreases exponentially [12] and the channels degenerate into straight lines. The channels partition the phase space into cells. The centers of the cells are stable fixed points, and surrounding them are local KAM regions. Passing through each unstable fixed point, there is one stable invariant manifold and one unstable invariant manifold of the fourth order map M^4. Useful

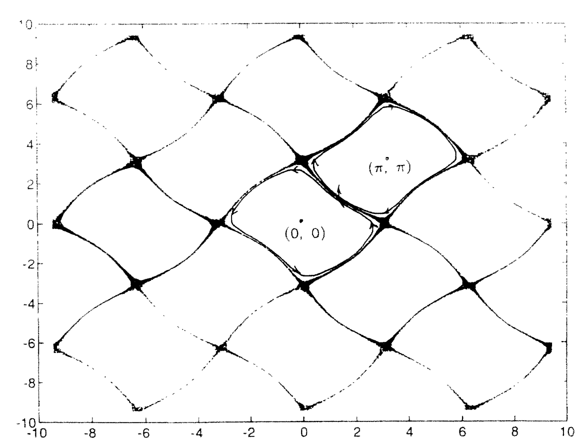

FIG. 1. Partition of phase space into cells and an illustration of particles' motion.

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information on properties of the manifolds can be found in Lowenstein's work [13]. The precise boundary between cells consists of curves formed by joining the manifolds at their primary intersection points [14]. With this choice of boundaries, an orbit of M^4 can cross over the boundary only in the region near the intersection point where a so-called turnstile is formed [14], and the orbit will not return to its previous cell before it finishes one revolution in the new cell.

Zaslavsky and co-workers have investigated various aspects of this web map, including the diffusion exponent [15–17]. Their numerical results show the likelihood of subdiffusion for small a. Lichtenberg and Wood have also studied diffusion of this map for small-a values [18]. Their numerical results show two interesting phenomena: average revolution numbers per boundary crossing noticeably depend on the length of orbits on which statistics are made, even though the lengths are of the order of 10^5 iterations; the root-mean square of particle distribution obtained by directly iterating orbits is about 1.5 to 2 times the value derived from iteration number per boundary crossing. They have correctly attributed these to long trapping by the KAM boundaries. They have also correctly pointed out that the long trapping tends to increase the rate of intermediate-time-scale diffusion over the asymptotic value. Though the authors have presumed somehow the diffusion is normal, and the agreement is reasonably good, the possibility that the diffusion is actually subdiffusive is not ruled out. In the present work, I will carefully investigate the long trapping, then I will set up a quantitative relationship between the long trapping and "anomalies" in diffusion statistics. I shall demonstrate rigorously that, due to characteristics of the power exponent of trapping time statistics of KAM boundaries, asymptotic subdiffusion is impossible, but in the case of weak chaos, there is a long subdiffusive transient, and this explains the apparent anomalies in the above mentioned works. Based on this work, I conclude that subdiffusion cannot be found in area-preserving mapping with globally uniform stochastic structure.

Besides the 2π translational symmetries in both x and y directions, the fourth iterate of the map commutes with a linear operator $L(x,y) = (\pi - x, \pi + y)$ and the inversion operator. From these, we know that the cell centered at $(0,0)$ and that at (π,π) are equivalent [13]. The phase space can be tiled by using one elementary cell. From symmetry consideration, one can derive that a particle should have an equal probability to exit from any side of the elementary cell. For $a = 0.3$, out of 477 987 events, there are 119 484, 119 508, 119 630, and 119 365 events exiting from the four different sides, respectively. Their deviations from $N = 1/4 \times 477\,987$ are -12.75, 11.25, 133.25, and -131.75, respectively; all of them are within $\sqrt{N} = 345.683$.

Since the centers of the cells form a square lattice of width $\sqrt{2}\pi$, I view the motion of the particle as follows. The particle is making a random walk on the lattice. It stays at a site for a time randomly selected out of a distribution function and then jumps to one of its nearest neighbors. Let τ, $\phi(\tau)$ denote the pause time and its distribution, respectively. Figure 2 shows the cumulative survival time distribution for $a = 0.3$, defined by $\xi(t) = \int_t^\infty \phi(\tau)d\tau$. We see that the asymptotic decay is faster than t^{-2}, so the average time spent in a cell exists. Define $P(n,t)$ as the probability that the particle has moved just n steps in time t. The correlation of exit direction can be regarded as independent of t and n because the probability of strongly correlated events is small; it is found to be 0.137.

Let $\mathcal{L}$ denote the Laplace transformation with the transform $\tilde{f}(u)$ of a function $f(t)$ defined as

$$\tilde{f}(u) = \mathcal{L}(f(t)) = \int_0^\infty e^{-ut} f(t)dt.$$

Since e^{-ut} is a rapidly decreasing factor, the large-t behavior of a function is reflected in the corresponding small-u behavior of its Laplace transformation.

One can express the second moment of x as

$$\langle x^2(t)\rangle = \sum_{n=0}^{\infty} \frac{1}{2}[1 + c(n,t)](\sqrt{2}\pi)^2 nP(n,t),$$

where $c(n,t)$'s are the correlation terms. Since empirically they are small and quite independent of n and t, I set it equal to a constant ϵ. Taking the Laplace transformation of the above equation,

$$\langle \tilde{x}^2(u)\rangle = (1+\epsilon)\pi^2 \sum_{n=0}^{\infty} n\tilde{P}(n,u). \qquad (1)$$

Since

$$P(n,t) = \int_0^t d\tau P(n-1, t-\tau)\phi(\tau),$$

one has

$$\tilde{P}(n,u) = \frac{1}{u}[1 - \tilde{\phi}(u)]\tilde{\phi}^n(u).$$

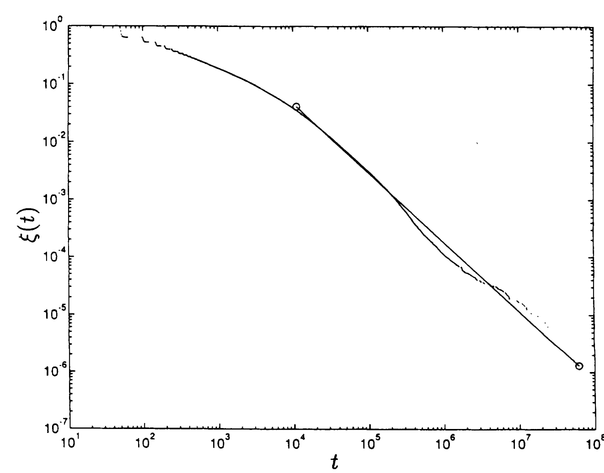

FIG. 2. Survival time distribution for $a = 0.3$. The ending part decays as $t^{-1.2}$.

Combining this with Eq. (1), one gets

$$\langle \tilde{x}^2(u) \rangle = (1+\epsilon)\pi^2 \frac{1}{u} \frac{\tilde{\phi}(u)}{1-\tilde{\phi}(u)}. \quad (2)$$

When $\phi(t)$ decays faster than t^{-2},

$$\left.\frac{d\tilde{\phi}(u)}{du}\right|_{u=0} = -\int_0^\infty t\phi(t)dt = -\overline{\tau}.$$

So, since

$$1-\tilde{\phi}(u) \simeq \overline{\tau}u,$$

$$\langle \tilde{x}^2(u) \rangle \simeq (1+\epsilon)\pi^2 \frac{1}{\overline{\tau}u^2},$$

one can get

$$\langle x^2(t) \rangle \simeq (1+\epsilon)\pi^2 \frac{t}{\overline{\tau}}.$$

This corresponds to normal diffusion.

If for large t

$$x^2(t) \sim t^\alpha,$$

with $0 < \alpha < 1$, then for small u,

$$\langle \tilde{x}^2(u) \rangle \simeq u^{-(1+\alpha)},$$

$$\tilde{\phi}(u) \simeq 1 - \text{const} \times u^\alpha.$$

This means for large t, $\phi(t) \sim t^{-1-\alpha}$, $\xi(t) \sim t^{-\alpha}$. In this case $\overline{\tau}$ does not exist.

Numerical experiments by different authors [8,19,20] for a variety of maps show that the tail part of $\xi(t)$ is usually between $t^{-1.3}$ and $t^{-1.5}$. Meiss and Ott attribute this to universal behavior near the last KAM curve, with the universal properties controlling the power law and the power exponent [21]. More recently, Zaslavsky *et al.* have studied self-similarity of "islands-around-islands" structures. They use a model of a random walk on a self-similar cluster and renormalization approach to explain the phenomenon [7].

For the map I use, at $a = 0.3$, $\xi(t) \sim t^{-1.2}$, and so, asymptotically, the diffusion should be normal, $x^2(t)$ should approach $(1+\epsilon)\pi^2 t/\overline{\tau}$. Figure 3 shows the numerical result. We see that the approach to normal behavior is rather slow and there is a long transient regime. I will show that this is because the asymptotic decay of the trapping time distribution is only slightly bigger than the critical value.

The following work is to find the leading anomalous terms. To simplify the expression, I rescale the time and space scales, setting the precoefficient of Eq. (1) to 1, and the average time in a cell $\overline{\tau}$ to 1. From Eq. (2) one can get the deviation from the exact normal behavior $\langle x^2(t) \rangle = t$,

$$\mathcal{L}(x^2(t) - t) = \frac{1}{u^2}\left(\frac{1}{\tilde{\xi}(u)} - (1+u)\right).$$

Since

$$\tilde{\xi}(0) = \left.\frac{1-\tilde{\phi}(u)}{u}\right|_{u\to 0} = \overline{\tau} = 1,$$

and when $\mathrm{Re}(u) > 0$, $\tilde{\xi}(u) < 1$, one can get

$$\mathcal{L}(x^2(t) - t) = \frac{1-\tilde{\xi}(u)}{u^2} + \frac{[1-\tilde{\xi}(u)]^2}{u^2} + \cdots - \frac{1}{u}. \quad (3)$$

If for large t, $\xi(t) \simeq c_1 t^{-\beta-1}$ with $0 < \beta < 1$, then, for small u,

$$\tilde{\xi}(u) \simeq 1 - c_1\Gamma(-\beta)u^\beta.$$

Setting $c = c_1[-\Gamma(-\beta)]$, and keeping the leading terms (up to u), we obtain from Eq. (3) the expansion

$$\mathcal{L}(x^2(t) - t) = \frac{1}{u^2}(cu^\beta + c^2u^{2\beta} + c^3u^{3\beta} + c^4u^{4\beta} - u),$$

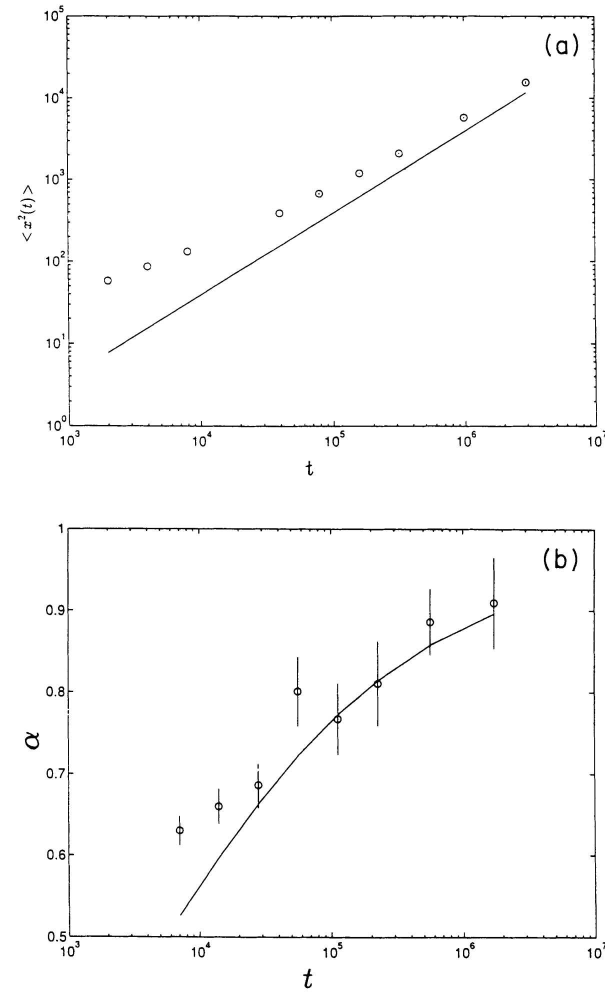

FIG. 3. Diffusion approaches normal behavior for $a = 0.3$. In both (a) and (b) open circles represent the numerical result. (a) $x^2(t)$ vs t. Error bars of the numerical values are smaller than the circle size. The straight line is given by $x^2(t) = (1+\epsilon)\pi^2 t/\overline{\tau}$. (b) α in $x^2(t) \sim t^\alpha$. The line is given by $\alpha = d\ln[x^2(t)]/d\ln(t)$ with the value of $x^2(t)$ obtained from Eq. (4).

which gives

$$\frac{x^2(t)-t}{t} \simeq \frac{c}{\Gamma(2-\beta)}t^{-\beta} + \frac{c^2}{\Gamma(2-2\beta)}t^{-2\beta} + \frac{c^3}{\Gamma(2-3\beta)}t^{-3\beta} + \frac{c^4}{\Gamma(2-4\beta)}t^{-4\beta} - t^{-1}. \tag{4}$$

The quantities c and β can be obtained from the survival time distribution. For $a = 0.3$ they are 1.19 and 0.203, respectively. Figure 4 is a check of Eq. (4). Considering there is no adjustable constant in Eq. (4), the agreement is good. All the above mentioned numerical studies were done for several other values of a in the range [0.25, 1.0]. The results are similar.

From Eq. (4), we see whenever c is large enough (of order 1), there should be a long intermediate anomalous regime which is thousands of times the average pause time. I have found for different maps (including the standard map) that whenever a KAM boundary is close to a low order unstable fixed point, the cumulative distribution function $\xi(t)$ decays slowly for time t in a range of a couple of orders of magnitude of the iteration number per revolution [18] of the boundary. This increases the weight of the tail. And thus for weak chaos c is usually large and so is the average pause time. A large anomalous time regime follows as a result.

Although the power index of the survival time distribution $\xi(t)$ cannot be accurately calculated [7,21], all signs show that it is greater than 1. Average trapping time near any KAM boundary exists and the correlation of the directions of walks leaving the boundary is short, so any area-preserving map with a macroscopically uniform stochastic structure should have asymptotically normal diffusion. Just like for the case of superdiffusion, the trapping power law is essential to a better understanding.

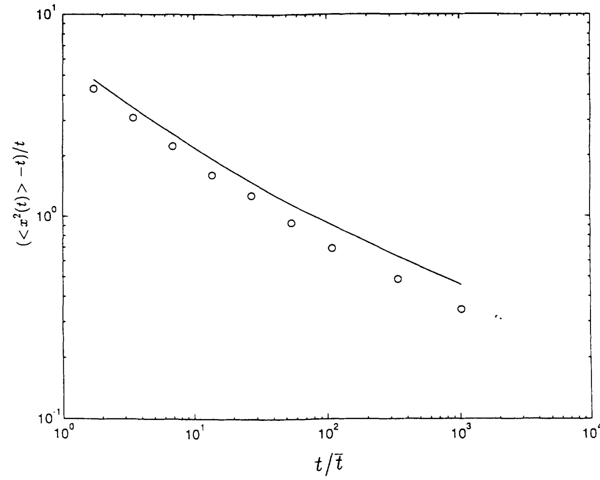

FIG. 4. Slowly decaying anomalous terms in $x^2(t)$. Open circles represent the numerical results. The line is given by Eq. (4).

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